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The aspects of reversible thinking in solving algebraic problems by an elementary student winning National Olympiad medals in science

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ABSTRACT: This study aimed to identify the aspects of reversible thinking of an elementary student who had won national Olympiad medals in science in solving particular algebraic problems. In this qualitative research, data were collected by a reversible thinking task contained a simple equation and an interview. The results showed that when making an equation, the only aspect of reversible thinking identified was reciprocity. This could be seen from the strategy the subject used in making the equation. It was done by dividing both sides of the equation by the same element, taking the square root of those two sides, and adding them with the same element. When reversing the new equation to its starting point, the aspects of reversible thinking identified were negation and reciprocity. It could be seen from the strategy the subject used in reversing the equation to its starting point. Negation was identified when the subject moved the parts of a known element, and in this case, the subject used a subtraction operation, which was the inversion of addition operation within the initial equation. Reciprocity was identified when the subject squared the both sides of the equation.

INTRODUCTION

This study was inspired by Piaget's theory on reversibility, that the primary characteristic of children aged seven up to 11 years of age was the development of a capacity for reversibility [1]. The researchers were also motivated by Lamon [2] and that the attention on reversibility was still limited. Hence, Lamon asked researchers in the education field to study students' reversibility. Krutetskii identified the mathematics skill related to the successfulness in solving problems, that is, reversibility and flexibility [3]. In fact, most primary graders found themselves having difficulties to solve arithmetical problems [4]. Inhelder and Piaget stated …*reversibility can be considered a key requirement in a number of problems in mathematics* [1], meaning that reversibility could be considered to be a primary requirement for solving a number of mathematics problems.

Ramful stated that *reversibility was related to mathematical operations, fractions, comparisons, algebra, and some other cases* [5]. Effective algebraic thinking sometimes involves reversibility (i.e. being able to undo mathematical processes, as well as to do them). In effect, it is the capacity not only to use a process to reach a goal, but also to be able to understand the process well enough to work backwards from the answer to the starting point. This definition indicated that it involved reversibility in solving algebraic problems, that is, an ability to reverse the mathematical process. Reversibility did not merely involve a process of achieving objectives, but also a process of reversing into the initial state with the generated answer. It suggested that there were two processes in reversibility: those that were a process of achieving objectives or a result, and a process of reversing the objectives or result to the initial state.

As Greenes said, algebra is sometimes referred to as generalised arithmetic, because it formalises arithmetic relationships [6]. Its power lies in the ways it allows one to represent relationships between quantities, to describe properties of operations (such as commutative and distributive), and to describe patterns. Algebra provides rules for manipulating symbols or signs, such as simplifying an expression and, then, solving the unknown. Suh stated the importance of algebra for elementary students, and asserting on *algebra-arithmetic* could be used to learn algebra concepts in elementary grades [7]. Algebra taught to elementary students was termed by Kaput as *early-algebra* [8]. It did not only facilitate the learning of the subsequent levels of algebra, but also helped to develop their mathematics concepts in a deeper and more complex manner since the early stage.

The subject of this present study was an elementary student who won National Olympiad medals in science, because a) based on Piaget's theory, the ability of reversible thinking begins to grow in a concrete operational phase in children from seven up to 11 years old [9]; b) the student who won National Olympiad medals in science was an asset of the nation that must be well-preserved, concerned and developed from early on; and c) the student who won National Olympiad medals in science had better mathematics skills than their peers, assuming that the subject could figure out reversible thinking in algebra. Therefore, this study aimed to identify the aspects of reversible thinking in solving particular algebraic problems by an elementary student who won the national Olympiad in science.

REVERSIBLE THINKING IN SOLVING ALGEBRAIC PROBLEMS

There were two important matters within reversible thinking, described as forward or reverse [1][3][10][11]. Forward was a mental process from a starting point that moved into the expected goal, whereas, reverse was a mental process from the expected goal moving to its starting point. The aspects of reversible thinking could be identified as follow:

Table 1: The aspects of reversible thinking that could be identified.

Aspect of reversible thinking	Explanation
Negation	It was when a subject used inversion towards the related operation [1][10][11].
Reciprocity	It was when a subject used compensation or any other equivalent relationships with a given equation [1][3][11].
Capability to return to the initial data after obtaining the result	It was when a subject could return the equation to its starting point using correct procedures [1][3].

Solving the problem was an activity of seeking solution for particular situation, using the insight previously obtained [12-14]. The focus of the study mentioned earlier was on algebra taught to elementary students. The algebra taught to them was referred to by the term *early-algebra* [8]. Early algebra was not merely a bridge to learning algebra at subsequent levels, but it also helped in developing the conception of students' mathematics in a deeper and more complex manner than earlier [6][8]. Greenes stated that in teaching mathematics for elementary graders, variables were used in three ways [6]. They could represent unknown elements, identify quantities that vary and generalise the properties. Powel stated that an equation is a mathematical statement in which the equal sign (=) was used to show the equivalency between a number or expression on the left-hand side and a number or expression on the right-hand side [14]. The expression itself was defined as a combination of operant numbers and arithmetic operations without any equal sign (=).

The problem of this study consisted of an initial equation containing one variable as an unknown element. The instruction provided was that the subject was asked to make as many equations as possible that were equivalent to the initial one. Thus, solving an algebraic problem was an activity the subject undertook in order to make as many equivalent equations as possible with its initial as a solution of the given problem using his/her insight. To identify the student's reversible thinking in solving algebraic problems, this study applied a test containing an equation. Then, the student was asked to make other equivalent equations based on the initial one. Two equations were defined as equivalent, if both had similar solutions. Thus, the indicators of reversible thinking that could be identified in algebra are included in Table 2.

Table 2: Indicators of the aspects of reversible thinking that could be identified in algebra.

Process of reversible thinking	Aspects of reversible thinking	Indicators
Forward (a process in which the subject made other equations equivalent with its initial)	Negation	When the subject used inversion towards the related operation in making equations.
	Reciprocity	When the subject used compensation or any other relationships equivalent with a given equation in making equations.
Reverse (a process in which the subject reversed the equations he just made into the initial one)	Negation	When the subject used inversion towards the related operation in his way reversing the equations.
	Reciprocity	When the subject used compensation or any other relationships equivalent with a given equation in reversing the equations.
	Capability to return to initial data after obtaining the result	When the subject could return the equation made to the initial one using correct procedures.

METHOD

Bogdan and Biklen explained the characteristics of qualitative research, and the present study is in accordance with these characteristics [15]. These are: a) naturalistic in nature, since it was conducted using the real situation as the data source and the researchers are its primary instrument; b) descriptive, since the data collected were qualitative, such as a set of words or writing, in cases when the data were in the form of the subject's work; and c) inductive, since it did not aim to prove any hypothesis, but merely to describe a phenomenon. The researchers provided a test for the subject, and conducted an interview later to reveal any other uncovered material relating to the test result. This study selected an elementary student who had won national Olympiad medals in science.

The procedures conducted in this study consisted of three primary phases as follows:

1. Preparation; in this phase, the researchers examined the theories of reversible thinking.
2. Implementation; in this phase, the researchers selected the subject for this study. Subsequently, the researchers gave a test to the subject and, then, conducted an interview based on their work.
3. Analysis; in this phase, the researchers conducted data analysis and wrote a report.

Analysis was conducted after the interview had ended. Subsequently, the researchers analysed the data entirely referring to the framework of reversible thinking, which is described in Table 2 using the following steps: 1) data reduction; 2) data presentation; and 3) conclusion making.

RESULTS AND DISCUSSION

The researchers initially engaged a group of elementary school students from which they selected a student who had won medals in the National Kuark Science Olympiad. Subsequently, the researchers conducted the research and analysed the data relating to the research results.

The test provided for the subject is presented in Figure 1.

TEST
Given the following form:
$$24 + a = 16$$
Make as many forms as possible based on the form above!

Figure 1: Test instrument.

The intended form of the test was an equation. The researchers used the term *form* since it was assumed that elementary students would not yet recognise the term *equation*. The work showed that the subject had successfully made 34 equations equivalent with the initial one. However, in this case, the researchers only analysed one of the 34 equations, as shown in Figure 2.

$$\sqrt{\frac{24+a}{4}} + 8 = 10$$

Figure 2: One of the equations made by the subject.

In order to identify reversible thinking, the researchers presented the procedures the subject applied in making equations based on the results of the interview. However, before presenting the result of the interview, the researchers initially described the code of the interview as follows:

PPi = i^{th} statement by the researchers
SJi = i^{th} answer by the subject

P1 : *Explain the procedures you used in making this form!* (by pointing out this $\sqrt{\frac{24+a}{4}} + 8 = 10$).

SJ1 : *Yes, the initial form is 24 added to a equals 16. Then, I divide both sides of this initial form by 4. The result is 24 added to a divided by 4 equals 4.*

P2 : *Would you please write it down?*

SJ2 : *Of course* (and, then, writing down the following form): $\frac{24+a}{4} = \frac{16}{4}$, then $\frac{24+a}{4} = 4$.

P3 : *And, then?*

SJ3 : *Then, I take the square root of both sides of this* (pointing out that $\frac{24+a}{4} = 4$) *and, the result will become...* (and writing down the following): $\sqrt{\frac{24+a}{4}} = \sqrt{4}$, then $\sqrt{\frac{24+a}{4}} = 2$.

P4 : *After that?*

SJ5 : *I add 8 to both sides of this* (pointing out $\sqrt{\frac{24+a}{4}} = 2$) *, so that it equals this…* (pointing out $\sqrt{\frac{24+a}{4}} + 8 = 10$).

P5 : *Is it OK, if I divide the right-hand side of the initial equation by 4 and the left-hand side by 2?*
SJ5 : *No, it is not allowed.*
P6 : *How come?*
SJ6 : *Because the arithmetical result of the right-hand side is different from the left-hand one.*
P7 : *Why it is not allowed for both sides to be different?*
SJ7 : *It is not allowed.*
P8 : *Why?*
SJ8 : *There is an equal sign* (what the subject indicated was …=…).

Based on the interview above, the aspects of reversible thinking were revealed in the procedures used to make equations equivalent with the initial one are presented in Table 3.

Table 3: Aspects of reversible thinking in making equations by the subject.

LMi [*)]		Activity of making equation	Aspects of reversible thinking revealed
LM1	:	The subject payed attention to the initial equation provided	-
LM2	:	The subject divided both sides of the initial equation (that was $24+a=16$) by 4, so that it resulted in $\frac{24+a}{4}=\frac{16}{4}$.	Reciprocity, since the subject used compensation, dividing both sides of the initial equation with 4.
LM3	:	Determining the result of operation of $\frac{16}{4}$, that was 4, so that the result would be $\frac{24+a}{4}=4$.	-
LM4	:	The subject squared the both sides of $\frac{24+a}{4}=4$, so that it resulted in $\sqrt{\frac{24+a}{4}}=\sqrt{4}$.	Reciprocity, since the subject used compensation, squaring both sides of $\frac{24+a}{4}=4$.
LM5	:	Determining the result of $\sqrt{4}$, which was 2, so that it resulted in $\sqrt{\frac{24+a}{4}}=2$.	-
LM6	:	The subject added the both sides of $\sqrt{\frac{24+a}{4}}=2$ with 8, so that it resulted in $\sqrt{\frac{24+a}{4}}+8=2+8$.	Reciprocity, since the subject used compensation, adding the both sides of $\sqrt{\frac{24+a}{4}}=2$ with 8.
LM7	:	Determining the result of $2+8$, which was 10, so that it resulted in $\sqrt{\frac{24+a}{4}}+8=2+8=10$.	-

[*)] LMi is i^{th} step in making equation

Subsequently, in order to identify the subject's reversible thinking in reversing the equations made into the initial one or its starting point, the researchers initially presented the procedures the subject applied to reverse the equations made into its starting point based on the following interview result.

P9 : *Now, please explain the procedure you used to reverse all these forms you had made* (by pointing out $\sqrt{\frac{24+a}{4}}+8=10$) *into its initial one or its starting point! But, would you please first, write down the new procedures before giving some explanation?*
SJ9 : *Yes, of course* (and, then, the subject wrote the following).

$\sqrt{\frac{24+a}{4}}+8=10$
$\sqrt{\frac{24+a}{4}}=10-8=2$, then $\frac{24+a}{4}=4$
$24+a=4\times4=16$.

P10 : *Now, would you please explain it to me!*
SJ10 : *OK, I moved this 8 into here* (what the subject intended was the left-hand side), *so that it resulted in 10 minus 8, which is equal to 2* (by pointing out $\sqrt{\frac{24+a}{4}}=10-8=2$).
P11 : *And, then?*
SJ11 : *Then, I squared both sides resulting in 24, added by* a *and divided by 4. The result was 4* (what he intended was $\frac{24+a}{4}=4$).
P12 : *And, then?*
SJ12 : *I multiplied both sides of this* (by pointing out $\frac{24+a}{4}=4$) *by 4, which was equal to 24, added a, which then equalled 16.*

Based on the interview above, the aspects of reversible thinking, revealed in the procedures the subject used to reverse the equations and constructed the initial one or went back to its starting point, are presented in Table 4.

Table 4: Aspects of reversible thinking in reversing equations by the subject.

LKi *)		Activity of reversing equation	Aspects of reversible thinking revealed
LK1	:	The subject paid attention to the equations made, that was $\sqrt{\frac{24+a}{4}}+8=10$.	-
LK2	:	The subject moved the side of element 8, so that it resulted in $\sqrt{\frac{24+a}{4}}=10-8$, then $\sqrt{\frac{24+a}{4}}=2$.	Negation, since the subject used inversion of the related operation, cancelling the addition of 8 (that was + 8) by subtracting 8 (that was -8) in which the subtraction operation was the inversion of the addition operation.
LK3	:	The subject determined the result of operation $10-8$, which was 2, so that it resulted in $\sqrt{\frac{24+a}{4}}=2$.	-
LK4	:	Squaring both sides of $\sqrt{\frac{24+a}{4}}=2$, so that it resulted in $\frac{24+a}{4}=2^2$.	Reciprocity, since the subject used compensation, squaring both sides of $\sqrt{\frac{24+a}{4}}=2$.
LK5	:	Determining the result of 2^2, which was 4, so that it resulted in $\frac{24+a}{4}=4$.	-
LK6	:	Multiplying both sides of $\frac{24+a}{4}=4$ by 4, which resulted in $24+a=16$.	Reciprocity, since the subject used compensation, multiplying both sides of $\frac{24+a}{4}=4$ by 4.

*) LKi is *i*th step in reversing the equation the subject made into its starting point

Based on Table 3 and Table 4, there were three matters the researchers were concerned with relating to reversible thinking when the subject made the equation. Those were on the steps with codes LM2, LM4 and LM6. There were other three matters the researchers considered when the subject reversed the new equation. Those were the steps with codes LK2, LK4 and LK6. In order to facilitate understanding, these codes are described in Table 5.

Table 5: Interesting matters on the steps of making and reversing a new equation into its initial form.

	Code	Interesting matter when the subject made equations	Code	Interesting matter when the subject reversed a new equation to its starting point
*)	LM2	Dividing both sides of an initial equation by 4.	LK2	Moving 8 from one side to the other side.
**)	LM4	Squaring both sides of the equation.	LK4	Squaring both sides of the equation.
***)	LM6	Adding 8 to both sides of the initial equation.	LK6	Multiplying both sides of the new equation by 4.

The explanation of Table 5:

*) In the 2nd step of making the equation (coded as LM2), the subject *divided* the sides of the initial equation by 4; however, when reversing the equation on 6th step (coded as LK6), the subject *multiplied* both sides of the new equation by 4.

**) In the 4th step in making the equation (coded as LM4), the subject took the square root of both sides of $\frac{24+a}{4}=4$, so that it resulted in $\sqrt{\frac{24+a}{4}}=2$, however, when reversing the equation in the 4th step (coded as LK4), the subject took the square root of both sides of $\sqrt{\frac{24+a}{4}}=2$; hence, it resulted in $\frac{24+a}{4}=4$.

***) In the 6th step in making the equation (coded as LM6), the subject *added* 8 to both sides of the equation $\sqrt{\frac{24+a}{4}}=2$; hence, it resulted in $\sqrt{\frac{24+a}{4}}+8=10$; however, when reversing the equation in the 2nd step (coded as LK2), the subject *moved* 8 from one side of the equation to the other, so that it resulted in $\sqrt{\frac{24+a}{4}}=10-8=2$.

The result showed that the reversible thinking of an elementary student who had won national Olympiad medals in science in solving particular algebraic problem had progressed beyond their age. It is not in accordance with the stage of cognitive development presented by Piaget [1]. The subject fully understood the equal sign (=). In accordance to the subject, the equal sign indicated that …*the both sides are equal or the right-hand side equals the left one*. Other students in the same developmental stage as the subject assumed that the equal sign (=) indicated the answer or the result of an operation. McNeil et al argued …*equal signs were often presented in standard operations-equals-answer contexts (e.g., 3 + 4 = 7) and were rarely presented in nonstandard operations on both sides contexts (e.g. 3 + 4 = 5 + 2)* [16]. Students often defined the equal sign (=) as the context of an answer. They seldom defined it as a link of two sides (right and left sides).

CONCLUSIONS

The aspects of reversible thinking in solving algebraic problems by an elementary student who had won National Olympiad medals in science were negation and reciprocity. When making an equation, the only aspect of reversible thinking identified was reciprocity.

This could be seen from the strategy the subject used in making the equation. It was by dividing both sides of equation with the same element, subject took the square root of those two sides, and adding to both the same element. Whereas, when reversing the new equation to its starting point, the aspects of reversible thinking identified were negation and reciprocity.

It could be seen from the strategy the subject used in reversing the equation that it was made into the starting point. Negation was identified when the subject moved the parts of the known element and, in this case, the subject used a subtraction operation, which was the inversion of an addition operation within the initial equation. On the other hand, reciprocity was identified when the subject took the square root of squared both sides of the equation and multiplied them.

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